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Post-Newtonian Estimation in Relativistic Optics

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A post-Newtonian analysis of the theory of gravity based on the metric $g_{ij}(x, y) = \gamma_{ij}(x) + \alpha/c^2(1 - 1/n^2)\, y_i y_j$ with the index of refraction $n(x, y)$ is given. A generalized Lagrange space endowed with this metric is used for the study of gravitational phenomena. The index of refraction $n(x, y)$ is expanded in integer powers of the gravitational potential $U = GM/rc^2$ and v^2/c^2. It is shown that solar system tests impose a constraint on a combination of the constant α, the post-Newtonian parameters defining the index of refraction $n(x, y)$, and the post-Newtonian parameter β associated to the Riemannian metric $\gamma_{ij}(x)$.

1. INTRODUCTION

The generalized Lagrange spaces (Kawaguchi and Miron, 1989) endowed with the metric

$$g_{ij}(x, y) = \gamma_{ij}(x) + \frac{\alpha}{c^2} y_i y_j, \qquad i, j = 1, 2, \ldots, n \tag{1}$$

where $\gamma_{ij}(x)$ is a Riemannian metric, y_i is the Liouville vector field, and α is a constant, have been used for the study of gravitational phenomena (Asanov and Kawaguchi, 1990; Roxburgh, 1990). The metric (1) was studied by Beil (1987, 1989) and used in some problems from electrodynamics. It is related to a new class of Finsler metrics (Beil, 1989). The post-Newtonian orbits for a theory of gravity based on the metric (1) are examined by Asanov and Kawaguchi (1990), who concluded that the observations of planetary motion impose the constraint $\alpha \leq 10^{-3}$. The model was reexamined by Roxburgh (1990), and it was shown that solar system tests do not impose a restriction on the value of α, but only on a combination of α and the standard post-Newtonian parameter β for a Riemannian metric.

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A generalization of the metric (1) has been considered by Kawaguchi (1991) in the form

$$g_{ij}(x, y) = \gamma_{ij}(x) + \frac{\alpha}{c^2}\left(1 - \frac{1}{n^2}\right) y_i y_l \tag{2}$$

where $n = n(x, y)$ is the index of refraction of the medium. This metric appears for the first time in Synge (1966) and it has been used in the study of the propagation of the electromagnetic waves in a medium with the index of refraction $n(x, y)$. A study of this metric from a geometrical point of view was done by Miron and Kawaguchi (1991*a*,*b*) with the main emphasis on applications to relativistic geometrical optics.

In this paper we present a post-Newtonian analysis of the theory of gravity based on the metric (2). We expand n^2 in integer powers of the gravitational potential $U = GM/rc^2$ and v^2/c^2:

$$n^2 = 1 + \varepsilon U + \delta \frac{v^2}{c^2} + \mu U \frac{v^2}{c^2} + \nu U^2 + \sigma \frac{v^4}{c^4} + \cdots \tag{3}$$

where ε, δ, μ, ν, and σ are new post-Newtonian parameters of the model. This choice is in accord with Fock's results (Fock, 1962) obtained from the study of light bending in a gravitational field. We show that solar system tests impose a constraint on a combination of α and the parameters β, ε, μ, ν, and σ, where β is the standard post-Newtonian parameter for a Riemannian metric (Will, 1986).

2. GENERALIZED LAGRANGE SPACE AND METRIC

We follow here the terminology in the book of Miron and Anastasiei (1987). Let M be a C^∞ m-dimensional real manifold (in particular we will choose $m = 4$), π: $TM \to M$ the tangent bundle of M, and (x^i, y^i) $(i, j, k, \ldots = 1, \ldots, m)$ the local coordinates on the total space TM. Suppose that $\gamma_{ij}(x)$, $x \in M$, is a pseudo-Riemannian metric on the base manifold M. Then, for a point $u \in TM$, with $\pi(u) = x$, $\gamma_{ij}(\pi(u))$ give us a d-tensor field on TM, symmetric, covariant of second order, and of rank m. Therefore, $y_i = \gamma_{ij}(x) y^j$ is a d-covector field on $\widetilde{TM}$. We denote

$$\|y\|^2 = \gamma_{ij}(x) y^i y^j \tag{4}$$

and consider the differentiable manifold $\widetilde{TM} = TM \setminus \{0\}$, where $\{0\}$ is the null section of the projection π: $TM \to M$. Consequently $\|y\|^2 \neq 0$ on $\widetilde{TM}$.

Assume that there is given a positive function $n(x, y)$ on $\widetilde{TM}$ and take

$$u(x, y) = \frac{1}{n(x, y)} \tag{5}$$

The function $n(x, y)$ is called the index of refraction. Then we consider

$$g_{ij}(x, y) = \gamma_{ij}(x) + [1 - u^2(x, y)]y_i y_j \tag{6}$$

The following properties can be verified:

(i) $g_{ij}(x, y)$ is a d-tensor field on $\widetilde{TM}$, covariant of second order, and symmetric.

(ii) $\operatorname{rank} \| g_{ij}(x, y) \| = m$.

The pair $M^m = (M, g_{ij}(x, y))$ is a generalized Lagrange space whose fundamental tensor (or metric tensor) is $g_{ij}(x, y)$. If $1/n^2 = 1 - \alpha/c^2$, then the metric $g_{ij}(x, y)$ reduces to the metric (1). On the other hand, the value $n(x, y) = 1$ implies that M^m coincides with a Riemannian space $V^m = (M, \gamma_{ij}(x))$.

Let us suppose now that on the manifold M there is a C^∞ nonnull vector field $V^i(x)$, $x \in M$. Then, it can be shown (Miron and Anastasiei, 1987) that the mapping S_v: $M \to TM$ given by

$$x^i = x^i, \qquad y^i = V^i(x), \qquad x \in M, \quad i = 1, \ldots, m \tag{7}$$

is a cross section of the projection π: $TM \to M$. Consequently, the section $S_v(M)$ is a submanifold in $\widetilde{TM}$.

The restriction to the section $S_v(M)$ of the fundamental tensor $g_{ij}(x, y)$ of the generalized Lagrange space M^m is the tensor field $g_{ij}(x, V(x))$ given by

$$g_{ij}(x, V(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, V(x))}\right) V_i V_j \tag{8}$$

where

$$V_i(x) = \gamma_g i_j(x) V^j(x) \tag{9}$$

This is just the metric previously considered by Synge (1966).

The triplet $\mathcal{M} = [M, V(x), n(x, V(x))]$ is called a *dispersive medium*. If $\partial n / \partial y^i = 0$, then $\mathcal{M}$ is called a nondispersive medium. The restriction of the generalized Lagrange space M^m to the section $S_v(M)$ is called the geometrical model of the dispersive medium $\mathcal{M}$ endowed with the Synge metric (Synge, 1966).

3. THE LAGRANGIAN OF THE MODEL

We will make now a post-Newtonian analysis of the theory of gravity based on the metric (2). We choose the Lagrangian in the form

$$L = -m_0 c (g_{ij} \dot{x}^i \dot{x}^j)^{1/2} \tag{10}$$

where m_0 is the mass of the test particle. Then, using the metric (2), this Lagrangian can be written as

$$L = -m_0 c \left\{ S \left[1 + \frac{\alpha}{c^2} \left(1 - \frac{1}{n^2} \right) S \right] \right\}^{1/2} \tag{11}$$

where

$$y_i = \gamma_{ij} \dot{x}^j \tag{12}$$

and

$$S = \gamma_{ij} \dot{x}^i \dot{x}^j \tag{13}$$

Considering M a 4-dimensional pseudo-Riemannian manifold and making the index convention $i = (0, a)$, $a = 1, 2, 3$, we have

$$S = \left(\gamma_{00} + 2\gamma_{0a} \frac{v^a}{c} + \gamma_{ab} \frac{v^a v^b}{c^2} \right) c^2 \tag{14}$$

where $v^a = dx^a/dt$ is the velocity vector. Then, introducing (14) in (11), we obtain

$$\begin{aligned} L = -m_0 c^2 \Bigg\{ & \gamma_{00} + 2\gamma_{0a} \frac{v^a}{c} + \gamma_{ab} \frac{v^a v^b}{c^2} + \alpha \left(1 - \frac{1}{n^2} \right) \\ & \times \left[\gamma_{00}^2 + 4\gamma_{0a}\gamma_{0b} \frac{v^a v^b}{c^2} + \gamma_{ab}\gamma_{cd} \frac{v^a v^b v^c v^d}{c^4} + 4\gamma_{00}\gamma_{0a} \frac{v^a}{c} \right. \\ & \left. + 2\gamma_{00}\gamma_{ab} \frac{v^a v^b}{c^2} + 4\gamma_{0a}\gamma_{bc} \frac{v^a v^b v^c}{c^3} \right] \Bigg\}^{1/2} \end{aligned} \tag{15}$$

For the pure Riemannian case $n = 1$ this Lagrangian reduces to the well-known expression

$$L_r = -m_0 c^2 \left(\gamma_{00} + 2\gamma_{0a} \frac{v^a}{c} + \gamma_{ab} \frac{v^a v^b}{c^2} \right)^{1/2} \tag{16}$$

4. THE LAGRANGIAN OF A STATIC GRAVITATIONAL FIELD WITH SPHERICAL SYMMETRY

For a static, spherically symmetric gravitational field we can choose (Asanov and Kawaguchi, 1990)

$$\gamma_{00} = 1 - 2U + 2\beta U^2 \tag{17}$$

$$\gamma_{ab} = -\delta_{ab}(1 + 2\gamma U) \tag{18}$$

$$\gamma_{0a} = 0; \qquad U = \frac{GM}{c^2 r} \tag{19}$$

Then, the Lagrangian (15) becomes

$$L=-m_0c^2\left\{1+\alpha\left(1-\frac{1}{n^2}\right)-2U\left[1+2\alpha\left(1-\frac{1}{n^2}\right)\right]\right.$$
$$-\frac{v^2}{c^2}\left[1+2\alpha\left(1-\frac{1}{n^2}\right)\right]-2U\frac{v^2}{c^2}\left[\gamma+2\alpha(\gamma-1)\left(1-\frac{1}{n^2}\right)\right]$$
$$\left.+2U^2\left[\beta+2\alpha(1+\beta)\left(1-\frac{1}{n^2}\right)\right]+\alpha\frac{v^4}{c^4}\left(1-\frac{1}{n^2}\right)\right\}^{1/2} \qquad (20)$$

where $v^2=\delta_{ab}v^av^b$ (δ_{ab} being the Kronecker symbol). This Lagrangian reduces to that of Asanov and Kawaguchi if

$$\frac{1}{n^2}=1-\frac{1}{c^2} \qquad (21)$$

i.e., for nondispersive media with constant index of refraction.

We choose then the expansion (3) for n^2, which for $\varepsilon=4$ and δ, μ, ν, $\sigma=0$ gives the expression

$$n^2=1+\frac{4GM}{c^2r} \qquad (22)$$

obtained by Fock (1962). This explains the expansion (3) previously considered.

Now, expanding the square root in (20) and omitting the terms m_0c^2, which do not contribute to the equations of motion, we obtain

$$\frac{L}{m_0}=c^2U\left(1-\frac{\alpha\varepsilon}{2}\right)+\frac{1}{2}v^2(1-\alpha\delta)$$
$$+U^2\left(\frac{1}{2}-\beta+\frac{3\alpha\varepsilon}{2}-\frac{\alpha\nu}{2}+\frac{\alpha\varepsilon^2}{2}+\frac{\alpha^2\varepsilon^2}{8}\right)c^2$$
$$+Uv^2\left(\frac{1}{2}+\gamma+\frac{3\alpha\varepsilon}{4}+\frac{3\alpha\delta}{2}+\varepsilon\alpha\delta+\frac{\alpha^2\varepsilon\delta}{4}-\frac{\alpha\mu}{2}\right)$$
$$+\frac{v^4}{c^2}\left(\frac{1}{8}+\frac{3\alpha\delta}{4}+\frac{\alpha\delta^2}{2}+\frac{\alpha^2\delta^2}{8}-\frac{\alpha\sigma}{2}\right) \qquad (23)$$

This expression reduces to a pure Riemannian one if $\alpha=0$ and ε, δ, μ, ν, $\sigma\neq 0$ or if ε, δ, μ, ν, $\sigma=0$ and $\alpha\neq 0$.

The Lagrangian (23) can be also written in the form

$$\frac{L}{m_0\lambda_0}=\frac{1}{2}v^2+Uc^2\frac{\lambda_1}{\lambda_0}+U^2c^2\frac{\lambda_2}{\lambda_0}+v^2U\frac{\lambda_3}{\lambda_0}+\frac{v^4}{c^2}\frac{\lambda_4}{\lambda_0} \qquad (24)$$

where

$$\lambda_0 = 1 - \alpha\delta$$

$$\lambda_1 = 1 - \frac{\alpha\varepsilon}{2}$$

$$\lambda_2 = \frac{1}{2} - \beta + \frac{3\alpha\varepsilon}{2} - \frac{\alpha\nu}{2} + \frac{\alpha\varepsilon^2}{2} + \frac{\alpha^2\varepsilon^2}{8} \tag{25}$$

$$\lambda_3 = \frac{1}{2} + \gamma + \frac{3\alpha\varepsilon}{4} + \frac{3\alpha\delta}{2} + \alpha\varepsilon\delta + \frac{\alpha^2\varepsilon\delta}{4} - \frac{\alpha\mu}{2}$$

$$\lambda_4 = \frac{1}{8} + \frac{3\alpha\delta}{4} + \frac{\alpha\delta^2}{2} + \frac{\alpha^2\delta^2}{8} - \frac{\alpha\sigma}{2}$$

This form of the Lagrangian is obtained by taking into account that the equations of motion do not change when the Lagrangian is multiplied by a constant.

Now, the approximation of zeroth order must coincide with Newtónian theory. Therefore, we must impose the constraint

$$\frac{\lambda_1}{\lambda_0} = 1 \tag{26}$$

which implies

$$\delta = \frac{\varepsilon}{2} \tag{27}$$

Using (27), we find that the Lagrangian (24) becomes

$$\frac{L}{m_0\lambda_0} = \frac{1}{2}v^2 + Uc^2 + \lambda_1' U^2c^2 + \lambda_2' v^2 U + \lambda_3' \frac{v^4}{c^2} \tag{28}$$

where

$$\lambda_1' = \frac{1/2 - \beta + 3\alpha\varepsilon/2 - \alpha\nu/2 + \alpha\varepsilon^2/2 + \alpha^2\varepsilon^2/8}{1 - \alpha\varepsilon/2}$$

$$\lambda_2' = \frac{1/2 + \gamma + 3\alpha\varepsilon/2 + \alpha\varepsilon^2/2 + \alpha^2\varepsilon^2/8 - \alpha\mu/2}{1 - \alpha\varepsilon/2} \tag{29}$$

$$\lambda_3' = \frac{1/8 + 3\alpha\varepsilon/8 + \alpha\varepsilon^2/8 + \alpha^2\varepsilon^2/32 - \alpha\sigma/2}{1 - \alpha\varepsilon/2}$$

For the Lagrangian (28) to support the perihelion shift test, it is necessary to impose the constraint (Will, 1986)

$$\begin{aligned}\Delta &= \lambda_1' + 2\lambda_2' + 4\lambda_3' \\ &= \frac{2 + 2\gamma - \beta + 6\alpha\varepsilon + 2\alpha\varepsilon^2 + \alpha^2\varepsilon^2/2 - \alpha\mu - 2\alpha\sigma - \alpha\nu/2}{1 - \alpha\varepsilon/2} \\ &= 3 \pm 10^{-2}\end{aligned} \tag{30}$$

In addition, for a self-consistent theory it is necessary that the metric (2) satisfy the light propagation condition. This means that we must impose the condition

$$\gamma_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\left[1 + \alpha\left(1 - \frac{1}{n^2}\right)\gamma_{km}\frac{dx^k}{dt}\frac{dx^m}{dt}\right] = 0 \tag{31}$$

Then, we must choose (Asanov and Kawaguchi, 1990; Roxburgh, 1990)

$$\gamma = 1 \tag{32}$$

Consequently, the constraint (30) becomes

$$\frac{4 - \beta + 6\alpha\varepsilon + 2\alpha\varepsilon^2 + \alpha^2\varepsilon^2/2 - \alpha\mu - 2\alpha\sigma - \alpha\nu/2}{1 - \alpha\varepsilon/2} = 3 \pm 10^{-2} \tag{33}$$

For a total dispersive medium we have $\varepsilon = \mu = \nu = 0$, and then the constraint (33) is simply

$$4 - \beta - 2\alpha\sigma = 3 \pm 10^{-2} \tag{34}$$

On the other hand, the nondispersive case means $\varepsilon = \mu = \sigma = 0$, and then the above constraint is

$$4 - \beta - \frac{\alpha\nu}{2} = 3 \pm 10^{-2} \tag{35}$$

The choosing of the parameters ε, δ, μ, ν, and σ in the expansion (3) of n^2 depends essentially on the physical nature of the dispersive medium. We distinguish the following two cases:

(i) If the medium $\mathcal{M}$ is nondispersive, then we must choose $\delta = \mu = \sigma = 0$ and this implies $\varepsilon = 0$. Therefore, in the case of nondispersive media the post-Newtonian parameters β, ν, and the constant α satisfy the constraint (35). We emphasize that this is the most frequent situation which appears in relativistic optics.

(ii) If the medium $\mathcal{M}$ is totally dispersive, i.e., the index of refraction depends only on velocity, $n = n(\dot{x})$, then we must choose $\varepsilon = \mu = \nu = 0$, which implies $\delta = 0$. Therefore, for such media the post-Newtonian parameters β, σ, and the constant α satisfy the constraint (34).

5. CONCLUSIONS

The analysis presented in this paper shows that for dispersive media with index of refraction $n = n(x, \dot{x})$ the solar system tests do not impose a restriction on the value of α, but only on a combination of α with the post-Newtonian parameter β and the parameters introduced in the expansion (3) of n^2. The constraint is given by the relation (33); of course, this constraint is complicated, the choice of the parameters in the expansion of n^2 depending on the physical nature of the space. The mentioned constraint simplifies essentially in the extreme cases of nondispersive media, when it has the simple form (35), and totally dispersive media, when it is given by (34).

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